Mathematical Olympiad

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1. An outward normal vector is taken on each side of the polygon. The length of the normal is equal to the length of corresponding side. Prove that the sum of these normal vectors is equal to the zero vector.

2. Give an example of an acute angle ϕ for which $\sin\phi$, $\cos\phi$ and $\tan\phi$ are rational numbers. Prove that there are infinitely many such angles.

3. Solve the equation
$$x\left(1+\frac{1}{\sqrt{x^2-1}}\right)=\frac{35}{12}.$$

4. Let $a_1, a_2, \ldots$ is arithmetic progression such that $a_1 = a^2$ and $a_{2022} = b^2$. Find
$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{2021}}+\sqrt{a_{2022}}}.$$

5. Compute the value of a_{2022} if $a_0 = 1$ and
$$a_{n+1}=\frac{a_n}{1+na_n},\quad n=0,1,2,\ldots$$

6. Find the limit for $0 < a < b$:
$$\lim_{n\to+\infty}\left(\frac{\sqrt[n]{a}-\sqrt[n]{b}}{2}\right)^n.$$

7. Is there a continuous function $f(x)$ for which $f(f(x)) = 1 - x^3$?

8. Compute the derivative $f^{(2022)}(0)$ of the function
$$f(x)=\frac{2^x-1}{2^x+1}.$$

9. Compute the integral
$$\int_0^{\pi/2}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx.$$

10. The number 6 can be obtained by multiplying its positive prime factors in two ways: $6 = 2 \cdot 3 = 3 \cdot 2$. It is not difficult to verify that the number 30 can be obtained in twelve ways: $30 = (2 \cdot 3) \cdot 5 = 2 \cdot (3 \cdot 5) = 3 \cdot (2 \cdot 5) = \ldots$ Check that the number 210 can be obtained in 120 ways: $210 = ((2 \cdot 3) \cdot 5) \cdot 7 = (2 \cdot 3) \cdot (5 \cdot 7) = \ldots$ How many ways can you get the number 881790? (This number is divided by 13, 17 and 19.)